Chapter 6

Sturm-Liouville Problems

Definition 6.1 (Sturm-Liouville Boundary Value Problem (SL-BVP)) *With the notation*

$$\mathcal{L}[y] \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y, \tag{6.1}$$

consider the Sturm-Liouville equation

$$\mathcal{L}[y] + \lambda r(x)\,y = 0, \tag{6.2}$$

where $p > 0$, $r \geq 0$, and p, q, r are continuous functions on interval $[a,\, b]$; along with the boundary conditions

$$a_1 y(a) + a_2 p(a) y'(a) = 0, \quad b_1 y(b) + b_2 p(b) y'(b) = 0, \tag{6.3}$$

where $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$.

The problem of finding a complex number μ if any, such that the BVP (6.2)-(6.3) with $\lambda = \mu$, has a non-trivial solution is called a Sturm-Liouville Eigen Value Problem (SL-EVP). Such a value μ is called an eigenvalue and the corresponding non-trivial solutions $y(.;\mu)$ are called eigenfunctions. Further,

(i) *An SL-EVP is called a* regular SL-EVP *if $p > 0$ and $r > 0$ on $[a,\, b]$.*

(ii) *An SL-EVP is called a* singular SL-EVP *if (i) $p > 0$ on $(a,\, b)$ and $p(a) = 0 = p(b)$, and (ii) $r \geq 0$ on $[a,\, b]$.*

(iii) *If $p(a) = p(b)$, $p > 0$ and $r > 0$ on $[a,\, b]$, p, q, r are continuous functions on $[a,\, b]$, then solving Sturm-Liouville equation* (6.2) *coupled with boundary conditions*

$$y(a) = y(b), \quad y'(a) = y'(b), \tag{6.4}$$

is called a periodic SL-EVP.

We are not going to discuss singular SL-BVPs. Before we discuss further, let us completely study two examples that are representatives of their class of problems.

6.1 Two examples

Example 6.2 *For $\lambda \in \mathbb{R}$, solve*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \tag{6.5}$$

For reasons that will be clear later on, it is enough to consider $\lambda \in \mathbb{R}$.

Case 1. *Let $\lambda < 0$. Then $\lambda = -\mu^2$, where μ is real and non-zero. The general solution of ODE in (6.5) is given by*

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \tag{6.6}$$

This y satisfies boundary conditions in (6.5) if and only if $A = B = 0$. That is, $y \equiv 0$. Therefore, there are no negative eigenvalues.

Case 2. *Let $\lambda = 0$. In this case, it easily follows that trivial solution is the only solution of*

$$y'' = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \tag{6.7}$$

Thus, 0 is not an eigenvalue.

Case 3. *Let $\lambda > 0$. Then $\lambda = \mu^2$, where μ is real and non-zero. The general solution of ODE in (6.5) is given by*

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{6.8}$$

This y satisfies boundary conditions in (6.5) if and only if $A = 0$ and $B\cos(\mu\pi) = 0$. But $B\cos(\mu\pi) = 0$ if and only if, either $B = 0$ or $\cos(\mu\pi) = 0$.

The condition $A = 0$ and $B = 0$ means $y \equiv 0$. This does not yield any eigenvalue. If $y \not\equiv 0$, then $b \neq 0$. Thus $\cos(\mu\pi) = 0$ should hold. This last equation has solutions given by $\mu = \frac{2n-1}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$. Thus eigenvalues are given by

$$\lambda_n = \frac{2n-1}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{6.9}$$

and the corresponding eigenfunctions are given by

$$\phi_n(x) = B\sin\left(\frac{2n-1}{2}x\right), \quad n = 0, \pm 1, \pm 2, \ldots \tag{6.10}$$

Note: All the eigenvalues are positive. The eigenfunctions corresponding to each eigenvalue form a one dimensional vector space and so the eigenfunctions are unique upto a constant multiple.

Example 6.3 *For $\lambda \in \mathbb{R}$, solve*

$$y'' + \lambda y = 0, \quad y(0) - y(\pi) = 0, \quad y'(0) - y'(\pi) = 0. \tag{6.11}$$

This is not a SL-BVP. It is a mixed boundary condition unlike the separated BC above. These boundary conditions are called periodic boundary conditions.

Case 1. *Let $\lambda < 0$. Then $\lambda = -\mu^2$, where μ is real and non-zero. In this case, it can be easily verified that trivial solution is the only solution of the BVP (6.11).*

Case 2. *Let $\lambda = 0$. In this case, general solution of ODE in (6.11) is given by*

$$y(x) = A + Bx \tag{6.12}$$

This y satisfies the BCs in (6.11) if and only if $B = 0$. Thus A remains arbitrary.

Thus 0 is an eigenvalue with eigenfunction being any non-zero constant. Note that eigenvalue is simple. An eigenvalue is called simple eigenvalue if the corresponding eigenspace is of dimension one, otherwise eigenvalue is called multiple eigenvalue.

Case 3. *Let $\lambda > 0$. Then $\lambda = \mu^2$, where μ is real and non-zero. The general solution of ODE in (6.11) is given by*

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{6.13}$$

This y satisfies boundary conditions in (6.11) if and only if

$$\begin{aligned} A\sin(\mu\pi) + B(1 - \cos(\mu\pi)) &= 0, \\ A(1 - \cos(\mu\pi)) - B\sin(\mu\pi) &= 0. \end{aligned}$$

This has non-trivial solution for the pair (A, B) *if and only if*

$$\begin{vmatrix} \sin(\mu\pi) & 1-\cos(\mu\pi) \\ 1-\cos(\mu\pi) & -\sin(\mu\pi) \end{vmatrix} = 0. \tag{6.14}$$

That is, $\cos(\mu\pi) = 1$. *This further implies that* $\mu = \pm 2n$ *with* $n \in \mathbb{N}$, *and hence* $\lambda = 4n^2$ *with* $n \in \mathbb{N}$.
Thus positive eigenvalues are given by

$$\lambda_n = 4n^2, \quad n \in \mathbb{N}. \tag{6.15}$$

and the eigenfunctions corresponding to λ_n *are given by*

$$\phi_n(x) = \cos(2nx), \quad \psi_n(x) = \sin(2nx), \quad n \in \mathbb{N}. \tag{6.16}$$

Note: All the eigenvalues are non-negative. There are two linearly independent eigenfunctions, namely $\cos(2nx)$ *and* $\sin(2nx)$ *corresponding to each positive eigenvalue* $\lambda_n = 4n^2$. *Compare these properties with that of previous example.*

6.2 Regular SL-BVP

We noted some properties of the SL-BVP Example 6.2. These properties hold for general Regular SL-BVPs as well.

Remark 6.4 *We record here some of the properties of regular SL-BVPs.*

(1) *The eigenvalues, if any, of a regular SL-BVP are real.*
PROOF :
Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of a regular SL-BVP and let y be corresponding eigenfunction. That is,

$$\mathcal{L}[y] + \lambda r(x)\, y = 0, \quad a_1 y(a) + a_2 p(a) y'(a) = 0, \quad b_1 y(b) + b_2 p(b) y'(b) = 0. \tag{6.17}$$

Taking the complex conjugates, we get

$$\mathcal{L}[\overline{y}] + \lambda r(x)\, \overline{y} = 0, \quad a_1 \overline{y}(a) + a_2 p(a) \overline{y}'(a) = 0, \quad b_1 \overline{y}(b) + b_2 p(b) \overline{y}'(b) = 0. \tag{6.18}$$

Multiply the ODE in (6.17) wtih $\overline{y}$, and multiply that of (6.17) with y, and subtracting one from the other yields

$$\left[p(y'\overline{y} - \overline{y}'y)\right]' + (\lambda - \overline{\lambda}) r y \overline{y} = 0 \tag{6.19}$$

Integrating the last equality yields

$$\left[p(y'\overline{y} - \overline{y}'y)\right]\Big|_a^b = -(\lambda - \overline{\lambda}) \int_a^b r(x)|y(x)|^2\, dx. \tag{6.20}$$

But LHS of the last equation is zero, since we have both $b_1 y(b) + b_2 p(b) y'(b) = 0$ and $b_1 \overline{y}(b) + b_2 p(b) \overline{y}'(b) = 0$, we also know that $b_1^2 + b_2^2 \neq 0$, and hence a certain determinant associated is zero.

Thus we have

$$(\lambda - \overline{\lambda}) \int_a^b r|y|^2\, dy = 0. \tag{6.21}$$

Since y, being an eigenfunction, is not identically equal to zero, and integral of non-negative function (since $r > 0$) is not zero, the only possibility is that $\lambda = \overline{\lambda}$. That is, λ is real. Note that we have used self-adjointness of operator $\mathcal{L}$ somewhere! ∎

(2) *The eigenfunctions of a regular SL-BVP corresponding to distinct eigenvalues are othogonal w.r.t. weight function r on $[a, b]$, that is, if u and v are eigenfunctions corresponding to distinct eigenvalues λ and μ respectively, then*

$$\int_a^b r(x)u(x)v(x)\,dx = 0. \tag{6.22}$$

PROOF :
As in the previous proof, writing down the equations satisfied by u and v, and multiplying the equation for u with v and vice versa, finally subtracting one from another, we get

$$\big[p(u'v - v'u)\big]' + (\lambda - \mu)ruv = 0 \tag{6.23}$$

Integrating the last equality yields

$$\big[p(u'v - v'u)\big]\Big|_a^b = -(\lambda - \mu)\int_a^b r(x)u(x)v(x)\,dx. \tag{6.24}$$

Reasoning exactly as in the previous proof, LHS of the above equality is zero. Since $\lambda \neq \mu$, we get the desired (6.22). ■

(3) *The eigenvalues of a regular SL-BVP are simple. Thus an eigenfunction corresponding to an eigenvalue is unique up to a constant multiple.*

PROOF :
Let ϕ_1 and ϕ_2 be two eigenfunctions corresponding to the same eigenvalue λ.
We recall from the section on Green's functions (the identity (5.44)) here:
By Lagrange's identity (5.20), we get $\frac{d}{dx}\big[p(\phi_1'\phi_2 - \phi_1\phi_2')\big] = 0$. This implies

$$p(\phi_1'\phi_2 - \phi_1\phi_2') \equiv c, \text{ a constant.} \tag{6.25}$$

Since ϕ_1 and ϕ_2 satisfy the boundary condition $\mathcal{U}_1[y] = 0$, we get the following

$$\begin{vmatrix} \phi_1(a) & \phi_1'(a) \\ \phi_2(a) & \phi_2'(a) \end{vmatrix} = 0. \tag{6.26}$$

Since $(\phi_1'\phi_2 - \phi_1\phi_2')$ is the wronskian of two solutions of a second order ODE, it is identically equal to zero. From here, it follows that ϕ_1 and ϕ_2 differ by a constant multiple. ■

In the above remark, we only analysed the properties of an eigenvalue or of two eigenfunctions corresponding to distinct eigenvalues. We have not proved the existence of eigenvalues for a regular SL-BVP so far. We are not going to do this, since the result follows easily from a much general theory of a subject known as *Functional analysis*, to be more specific, the topic is called *spectral theory*. Some references where a proof can be found are books on functional analysis by B.V. LIMAYE, YOSIDA and also books on ODE by CODDINGTON & LEVINSON, HARTMAN or even books on PDE by WEINBERGER. We will only state the result.

Theorem 6.5 *A self-adjoint regular SL-BVP has an infinite sequence of real eigenvalues $\big(\lambda_n\big)_{n\in\mathbb{N}}$, that are simple satisfying*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots \tag{6.27}$$

$$\text{with} \quad \lim_{n\to\infty} \lambda_n \to \infty. ■$$

Exercise 6.6 *Let $h > 0$ be a real number. Find eigenvalues and corresponding eigenvectors of the regular SL-BVP posed on the interval* $[0, 1]$

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + hy'(1) = 0.$$

6.3 Periodic SL-BVP

For a periodic SL-BVP also, eigenvalues are real, eigenfunctions corresponding to distinct eigenvalues are orthogonal w.r.t. weight function r, but eigenvalues need not be simple. We record these in the following remark.

Remark 6.7 *We record here some of the properties of periodic SL-BVPs.*

(1) *The eigenvalues, if any, of a regular SL-BVP are real.*

PROOF :

Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of a regular SL-BVP and let y be corresponding eigenfunction. That is,

$$\mathcal{L}[y] + \lambda r(x)\, y = 0, \quad a_1 y(a) + a_2 p(a) y'(a) = 0, \quad b_1 y(b) + b_2 p(b) y'(b) = 0. \tag{6.28}$$

Taking the complex conjugates, we get

$$\mathcal{L}[\overline{y}] + \lambda r(x)\, \overline{y} = 0, \quad a_1 \overline{y}(a) + a_2 p(a) \overline{y}'(a) = 0, \quad b_1 \overline{y}(b) + b_2 p(b) \overline{y}'(b) = 0. \tag{6.29}$$

Multiply the ODE in (6.28) wtih $\overline{y}$, and multiply that of (6.28) with y, and subtracting one from the other yields

$$\left[p(y'\overline{y} - \overline{y}'y)\right]' + (\lambda - \overline{\lambda}) r y \overline{y} = 0 \tag{6.30}$$

Integrating the last equality yields

$$\left[p(y'\overline{y} - \overline{y}'y)\right]\Big|_a^b = -(\lambda - \overline{\lambda}) \int_a^b r(x)|y(x)|^2\, dx. \tag{6.31}$$

But LHS of the last equation is zero, since both y and $\overline{y}$ satisfy the periodic boundary conditions. Note that this is new argument that we use to replace the correspondign argument in part (1) of Remark 6.4.

Thus we have

$$(\lambda - \overline{\lambda}) \int_a^b r|y|^2\, dy = 0. \tag{6.32}$$

Since y, being an eigenfunction, is not identically equal to zero, and integral of non-negative function (since $r > 0$) is not zero, the only possibility is that $\lambda = \overline{\lambda}$. That is, λ is real. Note that we have used self-adjointness of operator $\mathcal{L}$ somewhere! ■

(2) *The eigenfunctions of a periodic SL-BVP corresponding to distinct eigenvalues are othogonal w.r.t. weight function r on $[a, b]$, that is, if u and v are eigenfunctions corresponding to distinct eigenvalues λ and μ respectively, then*

$$\int_a^b r(x)u(x)v(x)\, dx = 0. \tag{6.33}$$

PROOF :

As in the previous proof, writing down the equations satisfied by u and v, and multiplying the equation for u with v and vice versa, finally subtracting one from another, we get

$$\left[p(u'v - v'u)\right]' + (\lambda - \mu) r u v = 0 \tag{6.34}$$

Integrating the last equality yields

$$\left[p(u'v - v'u)\right]\Big|_a^b = -(\lambda - \mu) \int_a^b r(x)u(x)v(x)\, dx. \tag{6.35}$$

Reasoning exactly as in the previous proof, LHS of the above equality is zero. Since $\lambda \neq \mu$, we get the desired (6.33). ■

(3) *The eigenvalues of a regular SL-BVP are not simple, and Example 6.3 illustrates this fact. However, it will be interesting to know why the arguments given in point (3) of Remark 6.4 can not be modified. In that proof we heavily rely on the form of boundary conditions for a regular SL-BVP, in deducing 6.26.* ■

In the above remark, we only analysed the properties of an eigenvalue or of two eigenfunctions corresponding to distinct eigenvalues. We have not proved the existence of eigenvalues for a periodic SL-BVP so far. We are not going to do this. We will only state the result, and you may refer to ODE book of Coddington & Levinson.

Theorem 6.8 *A self-adjoint periodic SL-BVP has an infinite sequence of real eigenvalues* $\left(\lambda_n\right)_{n\in\mathbb{N}}$ *satisfying*

$$-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots \tag{6.36}$$

The first eigenvalue λ_1 *is simple. The number of linearly independent eigenfunctions corresponding to any eigenvalue* μ *is equal to the number of times* μ *is repeated in the above listing.* ■

Exercise 6.9 *Find eigenvalues and eigenvectors of the following periodic SL-BVP posed on the interval* $[-\pi,\, \pi]$

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$